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Topology and its Applications 154 (2007) 404–409

Topology and its Applications

www.elsevier.com/locate/topol

Linearly ordered Radon–Nikodým compact spaces

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Received 5 March 2006; received in revised form 5 May 2006; accepted 5 May 2006

Abstract

We prove that every fragmentable linearly ordered compact space is almost totally disconnected. This combined with a result of Arvanitakis yields that every linearly ordered quasi-Radon–Nikodým compact space is Radon–Nikodým, providing a new partial answer to the problem of continuous images of Radon–Nikodým compacta.

MSC: 06A05; 54D30; 46B50; 46B26; 54D05

Keywords: Radon–Nikodým compact; Linear order; Almost totally disconnected space

It is an open problem posed by Namioka [8] whether the class of Radon–Nikodým compact spaces is closed under continuous images. Several authors [5,2], [1, p. 104] who have studied this problem have introduced some superclasses of the class of Radon–Nikodým compacta which are closed under continuous images, although all these classes turned out to be equal to the class of quasi-Radon–Nikodým compacta as shown in [9,3]. Let us recall that

(1) A compact space K is *Radon–Nikodým compact* if and only if there exists a lower semicontinuous metric $d : K \times K \to [0, +\infty)$ which fragments K.
(2) A compact space K is *quasi-Radon–Nikodým compact* if and only if there exists a lower semicontinuous quasi-metric $d : K \times K \to [0, +\infty)$ which fragments K.
(3) A compact space K is a *fragmentable compact* if and only if there exists a quasi-metric $d : K \times K \to [0, +\infty)$ which fragments K.

Here, a quasi-metric is a symmetric map $d : K \times K \to [0, +\infty)$ such that $d(x, y) = 0$ if and only if $x = y$ but which may fail triangle inequality. Also, a map $d : K \times K \to [0, +\infty)$ is said to fragment the topological space K if for every nonempty (closed) subset L of K and every $\varepsilon > 0$ there exists a relative open subset U of L of diameter less than ε, that is, $\sup\{d(x, y):\ x, y \in U\} < \varepsilon$. Lower semicontinuity means that the set $\{(x, y):\ d(x, y) \leqslant a\}$ is closed for every $a \geqslant 0$.

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[1] Author supported by FPU Grant and Grant MTM2005-08379 from MEC of Spain and Grant 00690/PI/2004 of Fundación Séneca of Región de Murcia.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.05.004

The class of fragmentable compacta is larger than the other two, for instance any Gul'ko non-Eberlein compact is an example of fragmentable and not quasi-Radon–Nikodým compact. It is again an open problem whether every quasi-Radon–Nikodým compact is Radon-Nikodým compact (as mentioned earlier, the class of quasi-Radon–Nikodým compacta is closed under continuous images, and it is even unknown whether every quasi-Radon–Nikodým compact is the continuous image of a Radon–Nikodým compact). Mainly two partial answers to this question are known:

(1) (Arvanitakis [2]) If K is an almost totally disconnected quasi-Radon–Nikodým compact, then K is Radon–Nikodým compact.
(2) (Avilés [3]) If K is a quasi-Radon–Nikodým compact of weight less than $\mathbf{b}$, then K is Radon–Nikodým compact.

In addition, we also mention that some conditions are given by Matoušková and Stegall [7] for a union of two Radon–Nikodým compacta to be Radon–Nikodým compact. In this paper we are mainly concerned with Arvanitakis' result, which generalizes previous work of [11,5]. We recall the concept of almost totally disconnected compact. We denote by $\Sigma[0,1]^\Gamma$ the set of all elements of $[0,1]^\Gamma$ with countable support and by $\Sigma_0^1[0,1]^\Gamma$ the subspace of $[0,1]^\Gamma$ formed by the elements such that all but countably many coordinates belong to $\{0,1\}$.

Definition 1. A compact space is said to be almost totally disconnected if it homeomorphic to a subset of $\Sigma_0^1[0,1]^\Gamma$ for some set Γ.

This class contains both the classes of Corson compacta (compact subsets of $\Sigma[0,1]^\Gamma$) and of totally disconnected compacta. On the other hand, an idea of the limitations of Arvanitakis' theorem may be suggested by the following remark, which shows that the class of almost totally disconnected spaces does not provide anything new with respect to Corson compacta, when we restrict our attention to path-connected compacta, and this includes the important case of probability measure spaces. We noticed this fact in conversation with Ondřej Kalenda during his visit to Murcia in March 2006.

Proposition 2. *Every path-connected almost totally disconnected compact space is a Corson compact.*

Proof. Take $K \subset \Sigma_0^1[0,1]^\Gamma$. We fix a point $x \in K$ and without loss of generality we shall suppose that $x_\gamma = 0$ for all but countably many γ's. In other words, there exists a countable set $\Gamma_x \subset \Gamma$ such that $x_\gamma = 0$ for every $\gamma \in \Gamma \setminus \Gamma_x$. Now we take any other point $y \in K$ and we shall check that y has also countable support. Since K is path connected, there is a separable and connected compact $L \subset K$ with $x, y \in L \subset K$. Let Q be a countable dense subset of L. For every $q \in Q$ there is a countable set $\Gamma_q \subset \Gamma$ such that $q_\gamma \in \{0,1\}$ for every $\gamma \in \Gamma \setminus \Gamma_q$. The set $\Gamma_L = \bigcup_{q \in Q} \Gamma_q$ is a countable subset of Γ such that $p_\gamma \in \{0,1\}$ for every $p \in L$ and every $\gamma \in \Gamma \setminus \Gamma_L$. Since L is connected, the set $\{p_\gamma\colon p \in L\}$ must be connected for every $\gamma \in \Gamma$. If we take $\gamma \in \Gamma \setminus (\Gamma_L \cup \Gamma_x)$ then $\{0\} \subset \{p_\gamma\colon p \in L\} \subset \{0,1\}$, so connectedness implies $\{p_\gamma\colon p \in L\} = \{0\}$. Applying this in particular to $p = y$, we found that $y_\gamma = 0$ whenever $\gamma \in \Gamma \setminus (\Gamma_L \cup \Gamma_x)$, so y has countable support. □

Apparently, we used a weaker hypothesis than path-connected in this result, namely that every two points are contained in a separable connected compact. However this is equivalent in this context, because a separable connected compact which is almost totally disconnected must be metrizable: Take $K \subset \Sigma_0^1[0,1]$ and suppose that $D \subset K$ is a countable dense subset of K. Then, we can find a countable subset $\Gamma' \subset \Gamma$ such that $D \subset [0,1]^{\Gamma'} \times \{0,1\}^{\Gamma \setminus \Gamma'}$. Hence $K \subset [0,1]^{\Gamma'} \times \{0,1\}^{\Gamma \setminus \Gamma'}$ and the connectedness of K implies that we have an embedding $K \hookrightarrow [0,1]^{\Gamma'}$, so K is metrizable. On the other hand, the assumption of being path-connected cannot be weakened to just being connected: several examples of connected almost totally disconnected compacta which are not Corson will be described below.

The following Theorem 3 is the main result of this note. Its proof is presented in Section 1.

Theorem 3. *Let K be a linearly ordered fragmentable compact. Then K is almost totally disconnected.*

Corollary 4. *Let K be a linearly ordered quasi-Radon–Nikodým compact. Then K is a Radon–Nikodým compact.*

A typical example of a linearly ordered compact which is not fragmentable is the split interval (also known as double-arrow space), that is, the set $K = [0,1] \times \{0,1\}$ ordered lexicographically. Indeed any variant of the split interval on which uncountably many points are splitted fails to be fragmentable. The reason is that if d is any quasi-metric on K, then there is an uncountable set $A \subset (0,1)$ and $\varepsilon > 0$ such that $d((x,0),(x,1)) > \varepsilon$ for every $x \in A$. If B is a subset of A in which every point of B is the limit of elements of B both from the right and from the left, then the set $B \times \{0,1\}$ fails to contain any relative open subset of diameter less than ε.

The class of linearly ordered compacta is a rather restrictive class of compact spaces. For example, it is a result of Efimov and Čertanov [4], with an alternative proof due to Gruenhage [6], that every linearly ordered Corson compact space is metrizable. In the view of this result and also of Proposition 2, one may be suspicious about real application of Theorem 3. This is not the case and indeed one of the examples of Radon–Nikodým compact proposed by Namioka [8] is the so-called extended long line. This is a linearly ordered compact obtained from the ordinals less or equal to ω_1 by inserting a copy of the interval $(0,1)$ between every two consecutive countable ordinals. More examples of linearly ordered Radon–Nikodým compacta are constructed in Section 2, where Corollary 4 will find application.

It is an open question for us whether every fragmentable linearly ordered compact must be a Radon–Nikodým compact.

1. Proof of the main theorem

We begin with a couple of lemmas, stating reformulations of the concept of almost totally disconnected compact, the second of them in the framework of linearly ordered compacta.

Lemma 5. *For a compact space K the following are equivalent*:

(1) *K is almost totally disconnected.*
(2) *There is a collection $\{(F_i,H_i)\}_{i\in I}$ of pairs of closed subsets of K such that*
   (a) *$F_i \cap H_i = \emptyset$ for all $i \in I$.*
   (b) *For any x in K, the set $\{i \in I\colon\ x \notin F_i \cup H_i\}$ is countable.*
   (c) *For any two different points x, y in K there is some $i \in I$ such that $x \in F_i$ and $y \in H_i$ or vice versa.*

Proof. For (1) $\Rightarrow$ (2), suppose $K \subset \Sigma_0^1[0,1]^\Gamma$. For each $\gamma \in \Gamma$ and each pair r, s of rational numbers with $0 \leqslant r < s \leqslant 1$, call $i = (\gamma, r, s)$,

$$F_i = \{x \in K\colon\ x_\gamma \leqslant r\},$$

$$H_i = \{x \in K\colon\ x_\gamma \geqslant s\}.$$

These (F_i,H_i) satisfy all desired conditions.

Conversely, suppose we are given a family like in (2). For each i, by Tietze's theorem, there is a continuous map $f_i : K \to [0,1]$ such that $f_i(F_i) = \{0\}$ and $f_i(H_i) = \{1\}$. In this case we have an embedding $f : K \to \Sigma_0^1[0,1]^I$ given by $f(x) = (f_i(x))_{i\in I}$. $\square$

Lemma 6. *Let $(K, \leqslant)$ be a linearly ordered compact space. The following are equivalent*:

(1) *K is almost totally disconnected.*
(2) *There is a collection $\{(a_i,b_i)\}_{i\in I} \subset K \times K$ such that*
   (a) *$a_i < b_i$ for every $i \in I$.*
   (b) *For all x in K, the set $\{i \in I\colon\ a_i < x < b_i\}$ is countable.*
   (c) *For all $x < y$ in K, there is some $i \in I$ such that $x \leqslant a_i < b_i \leqslant y$.*

Proof. Clearly (2) implies (1) because $F_i =]-\infty, a_i]$ and $H_i = [b_i, +\infty[$ satisfy the conditions of Lemma 5. Conversely, suppose that we have a family $(F_j,H_j)_{j\in J}$ of couples of closed subsets of K satisfying the conditions of Lemma 5. Take as $\{(a_i,b_i)\}_{i\in I}$ the set of all pairs in $K \times K$ such that

(1) $a_i < b_i$.
(2) There is some $j(i) \in J$ such that
    (a) Either $a_i \in F_{j(i)}$ and $b_i \in H_{j(i)}$, or vice versa, $b_i \in F_{j(i)}$ and $a_i \in H_{j(i)}$.
    (b) There is no $x \in F_{j(i)} \cup H_{j(i)}$ such that $a_i < x < b_i$.

First, we check that for every $x \in K$, the set $\{i \in I\colon a_i < x < b_i\}$ is countable. Notice that whenever $a_i < x < b_i$, then $x \notin F_{j(i)} \cup H_{j(i)}$ and we know that, since the family $\{(F_j, H_j)\}_{j \in J}$ satisfies condition (b) of Lemma 5, the set $\{j \in J\colon x \notin F_j \cup H_j\}$ is countable. The fact that $\{i \in I\colon a_i < x < b_i\}$ is countable follows now from the observation that whenever $j(i) = j(i')$ and $i \neq i'$, the intervals $]a_i, b_i[$ and $]a_{i'}, b_{i'}[$ are disjoint.

Second, we check condition (c) of the lemma. Take $x < y$. Since condition (c) of Lemma 5 is satisfied, we suppose that there is some $j \in J$ such that $x \in F_j$ and $y \in H_j$. Let $z = \max\{t \in F_j\colon t \leqslant y\}$ and $z' = \min\{t \in H_j\colon z \leqslant t\}$. Then, (z, z') equals some (a_i, b_i) and $x \leqslant a_i < b_i \leqslant y$.

We pass now to the proof of Theorem 3 itself. Let K be a linearly ordered compact and let d be a quasi-metric which fragments K. We construct our family $\{(a_i, b_i)\} \subset K \times K$ as in Lemma 6 as follows. First, let $\{(a_i, b_i)\}_{i \in I_0}$ be the set of all pairs of immediate successors (that is, all $a_i < b_i$ such that the open interval $]a_i, b_i[$ is empty). For $n \geqslant 1$, by virtue of Zorn's Lemma, we can choose a family $(a_i, b_i)_{i \in I_n}$ which is maximal for the following properties:

(1) $a_i < b_i$ for every $i \in I_n$.
(2) The d-diameter of the open interval $]a_i, b_i[$ is less than $1/n$.
(3) $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ whenever i, j are different indices in I_n.

We take $I = \bigcup_{n=0}^{\infty} I_n$ and $(a_i, b_i)_{i \in I}$ as the family required in Lemma 6. Condition (a) of Lemma 6 is clearly satisfied and condition (b) follows from property (3) in the definition of I_n. Only condition (c) needs to be checked. Take $x < y$, and we suppose that

(A) there is no index $i \in I$ such that $x \leqslant a_i < b_i \leqslant y$.

This implies, by the definition of I_0, that no immediate successors can occur between x and y. This means that the interval $[x, y]$ is connected (and so all its subintervals).

It is not possible that for all n, there is $j \in I_n$ such that $]x, y[\subset]a_j, b_j[$. This is because, by property (2) of I_n, the d-diameter of $]x, y[$ would be 0, which is a contradiction. Hence,

(B) for some fixed $n_0 \in \omega$, there is no $j \in I_{n_0}$ such that $]x, y[\subset]a_j, b_j[$.

Claim 1. *There exists $i_0 \in I_{n_0}$ such that either $x < a_{i_0} < y$ or $x < b_{i_0} < y$.*

Proof. If such an i_0 does not exist, assertion (B) implies that $]x, y[\cap]a_i, b_i[= \emptyset$ for all $i \in I_{n_0}$. Since d fragments K there is a nonempty open interval $]u, v[\subset]x, y[$ of d-diameter less than $\frac{1}{n_0}$. By passing to a subinterval (recall that all intervals are connected now) it can be supposed that even $[u, v] \subset]x, y[$ and then, the pair (u, v) could be added to the family $\{(a_i, b_i)\}_{i \in I_{n_0}}$ in contradiction with its maximality. □

Without loss of generality, we assume that $x < a_{i_0} < y$ in Claim 1.

Claim 2. *There exists $j_0 \in I_{n_0}$ such that $x < b_{j_0} < a_{i_0} < y$.*

Proof. Again, if such a j_0 does not exist, then $]x, a_{i_0}[\cap]a_j, b_j[$ is empty for all $j \in I_{n_0}$ and, because of the fragmentability condition, we can find an interval $[u, v] \subset]x, a_{i_0}[$ of d-diameter less than $\frac{1}{n_0}$. In this case, the couple (u, v) could be added to the family $\{(a_j, b_j)\}_{j \in I_{n_0}}$ in contradiction with its maximality. □

Claim 3. *There exists $k_0 \in I_{n_0}$ such that $x < b_{j_0} < a_{k_0} < b_{k_0} < a_{i_0} < y$.*

Proof. Similarly, if such a k_0 did not exist, then $]b_{j_0}, a_{i_0}[\cap]a_k, b_k[=\emptyset$ for all $k \in I_{n_0}$. Since d fragments K, there should be an interval $[u, v] \subset]b_{j_0}, a_{i_0}[$ of d-diameter less than $\frac{1}{n_0}$ which leads once again to a contradiction. □

Claim 3 is incompatible with assumption (A) and this finishes the proof of Theorem 3. □

Notice that we did not use the full strength of the definition of fragmentability. We just needed that $d(x, y) > 0$ if $x \neq y$ and that every *interval* contains an open subinterval of d-diameter less than ε for every $\varepsilon > 0$.

2. Examples of linearly ordered Radon–Nikodým compacta

We recall a different characterization of quasi-Radon–Nikodým compacta. A metric $d : K \times K \to [0, +\infty)$ on the compact space K is called a Reznichenko metric [1, p. 104] if for every two different points $x, y \in K$ there exist neighborhoods U and V of x and y respectively such that $\inf\{d(u, v):\ u \in U,\ v \in V\} > 0$. The following theorem is due to Namioka [9]:

Theorem 7. *A compact space K is quasi-Radon–Nikodým if and only if there exists a Reznichenko metric on K which fragments K.*

We present now a method for constructing linearly ordered Radon–Nikodým compact spaces inspired on Ribarska's characterization of fragmentability [10]. We consider $\{T_n:\ n = 1, 2, \ldots\}$ to be a sequence of well ordered sets such that $T_n \subset T_{n+1}$. Without loss of generality, we shall assume that all T_n's have the same minimum and the same maximum. Let T be the linearly ordered set $T = \bigcup_{n=1}^{\infty} T_n$ and let $\overline{T}$ be the completion of T (by the completion of T we mean the only linearly ordered set $\overline{T}$ such that $T \subset \overline{T}$, $\overline{T}$ is compact in the order topology and $]x, y] \cap T \neq \emptyset$ for every $x, y \in \overline{T}$, $x < y$). Then, $\overline{T}$ is a linearly ordered compact space and moreover:

Theorem 8. *The space $\overline{T}$ is Radon–Nikodým compact.*

This theorem produces different examples of linearly ordered Radon–Nikodým compacta depending on the growing sequence of well ordered sets $T_1 \subset T_2 \subset \cdots$ that we may take as a basis. Before passing to the proof, we shall have a look at how these different constructions may look like. Notice that $\overline{T}$ is connected whenever for all $x < y$ in T_n there exists $z \in T_{n+1}$ such that $x < z < y$.

- If $T_0 = \{0, 1\}$ and T_{n+1} is constructed by adding a single new point between every two consecutive elements of T_n, then $\overline{T} = [0, 1]$.
- If T_0 is the set of all ordinals which are less than or equal to ω_1 and again T_{n+1} is constructed by adding a single new point between every two consecutive elements of T_n, then $\overline{T}$ is the extended long line.
- If T_0 is the set of all ordinals which are less than or equal to ω_1 and T_{n+1} is constructed by adding a copy of the set of all countable ordinals between every two consecutive elements of T_n, then $\overline{T}$ has no metrizable open subsets, since every open interval contains a copy of ω_1.

Proof of Theorem 8. For $x, y \in \overline{T}$, $x < y$ we define:

$$d(x, y) = \frac{1}{\min\{n:\ ]x, y] \cap T_n \neq \emptyset\}}$$

and also $d(x, y) = 0$ if $x = y$, and $d(x, y) = d(y, x)$ if $x > y$. Observe that:

(1) Since $\overline{T}$ is the completion of T, if $x < y$ then $T \cap]x, y]$ is nonempty. This implies that $d(x, y)$ exists and is a positive real whenever $x \neq y$.
(2) An easy case-by-case consideration proves that d verifies triangle inequality: $d(x, z) \leqslant d(x, y) + d(y, z)$.
(3) The metric d is a Reznichenko metric, that is, every two different points have neighborhoods at a positive d-distance. Namely, if $x < y$ and there is some $z \in]x, y[$, then $]x, z] \cap T$ is nonempty and there is $u \in T_n \cap]x, z]$ for some n. In this case $]-\infty, u[$ and $]u, +\infty[$ are neighborhoods at least $\frac{1}{n}$-d-distance. The other possibility is that

$]x, y[$ is empty. Then $y \in T$ and hence, $y \in T_n$ for some n. In this case $(-\infty, x]$ and $[y, +\infty[$ are neighborhoods of x and y at least $\frac{1}{n}$-d-distance.

(4) The metric d fragments $\overline{T}$. Given L a closed subset of $\overline{T}$ of more than one point, and $n \in \omega$, let $x = \min(L)$ and $y = \min\{z \in T_n\colon z > x\}$. Then, $L \cap [x, y[$ is a nonempty relative open subset of L of d-diameter less than $\frac{1}{n}$.

It follows from Theorem 7 that K is quasi-Radon–Nikodým compact and hence, by Corollary 4, it is Radon–Nikodým compact. □

We point out that the metric d that we have defined is not lower semicontinuous except in trivial cases and hence, the use of Corollary 4 cannot be avoided unless we want to embark in the construction of a more complicated metric. In some cases, a nice lower semicontinuous fragmenting metric may be available, as for the extended long line [8].

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